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COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED METRIC SPACES AND APPLICATIONS

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ABSTRACT: The aim of this paper is to prove some coupled fixed point theorems for mapping having the mixed monotone property in partially ordered metric spaces, which are generalization of the main results of Bhaskar and Lakshmikantham [6], and Mizoguchi and Takahashis fixed point Theorem. In addition, the existence and uniqueness for solution of periodic boundary value problem (PBVP) are studied.

Keywords: Coupled fixed point, Complete metric space, Fixed point theorem, Partially ordered metric space.

INTRODUCTION

Ran and Reurings initiated studying of fixed-point property on partial ordered sets in [14]. Where gave many useful results in matrix equations. In establishing the existence of a unique solution to periodic boundary value problem, authors in [12, 13], extend results of [14], and applied them to the periodic boundary value problem, considering both monotone cases. Existence of a fixed point for contraction type mappings in partially ordered metric spaces and applications have been considered recently by many authors (for more details see, [1, 2, 4, 8, 9]).
In [6], Bhaskar and Lakshmikantham have introduced notions of a mixed monotone mapping and a coupled fixed point and proved some coupled fixed point theorems for mixed monotone mapping and discuss the existence and uniqueness of solution for periodic boundary value problem.

Following definition coincides with the notion of a mixed monotone function on $(\mathbb{R}^2,\leq)$, where $\leq$ is the usual total order in $\mathbb{R}$.

Definition 1.1

() *Let $(X,\leq)$ be a partially ordered set and $F: X\times X \to X$. Mapping F is said to be has the mixed monotone property if $F(x,y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for every $x, y \in X$,*

1. for each $x_1, x_2 \in X$, if $x_1 \leq x_2$, then $F(x_1, y) \leq F(x_2, y)$;
2. for each $y_1, y_2 \in X$, if $y_1 \leq y_2$, then $F(x, y_1) \geq F(x, y_2)$.

Let $(X,\leq)$ be a partially ordered set and d be a metric on X such that (X,d) is a complete metric space. The product space $X\times X$ is endowed with the following partial order:

$$\text{for} \quad (x,y),(u,v)\in X\times X, \quad (u,v)\leq(x,y) \Leftrightarrow x\geq u, \quad y\leq v.$$

Definition 1.2

() *Let $(X,\leq)$ be a partially ordered set and $F:X\times X\to X$. An element $(x,y)\in X\times X$ is said to be a coupled fixed point of the mapping F, if $F(x,y)=x$ and $F(y,x)=x$.*

Gnana Bhaskar and Lakshmikantham in [6], proved the following important Theorem:

Theorem 1.3

[6, Theorem 2.1]Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $F:X\times X\to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k\in[0,1)$ with

$$d(F(x,y),F(u,v))\leq\frac{k}{2}[d(x,u),d(y,v)],$$

for all $x\geq u$ and $y\leq v$. If there exist two elements $x_0,y_0\in X$ with

$$x_0\leq F(x_0,y_0)\quad\text{and}\quad y_0\geq F(x_0,y_0).$$

Then there exist $x,y\in X$ such that

$$x=F(x,y)\quad\text{and}\quad y=F(y,x).$$

Consistent with [11], let $CB(X)$ be the class of all nonempty bounded and closed subsets of X, and $K(X)$ be the class of all nonempty compact subsets of X. Let H be the Hausdorff metric on $CB(X)$ induced by the metric d of X and given by

$$H(A,B)=\max\{\sup_{x\in A}d(x,B),\sup_{y\in B}d(y,A)\}$$

for every $A,B\in CB(X)$. It is obvious that $K(X)\subseteq CB(X)$.

A point $x\in X$ is called a fixed point of a multivalued mapping $T:X\to CB(X)$ if $x\in Tx$. Reich in [15], proved that if (X,d) is a complete metric space and $T:X\to CB(X)$ satisfies

$$H(Tx,Ty)\leq\alpha(d(x,y))d(x,y),$$

for each $x,y\in X$, where $\alpha:[0,\infty)\to[0,1)$ such that $\limsup_{r\to t^+}\alpha(r)<1$ for each $t\in(0,\infty)$, then T has a fixed point. Reich raised the question of whether $K(X)$ can be replaced by $CB(X)$ in this result. In [10], Mizoguchi and Takahashi gave a positive answer to the conjecture of Reich. Du in [5], proved some coupled fixed point results of Mizoguchi and Takahashi's type in partially quasiordered metric spaces. Other version of Mizoguchi and Takahashi's fixed point Theorem considered by Amini-Harandi and O' Regan in [2].

Let Φ be the family of all functions $\varphi:[0,\infty)\to[0,\infty)$ satisfying the following conditions:

1. $\varphi(s)=0\Leftrightarrow s=0$;
2. φ is lower semicontinuous and nondecreasing;
3. $\limsup_{s\to 0}\dfrac{s}{\varphi(s)}<\infty$.

Recently in [7], Gordji and Ramezani generalized the Mizoguchi and Takahashi Theorem for single-valued mappings as follows:

Theorem 1.4

[7, Theorem 3.1]Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $f:X\to X$ be an increasing mapping such that there exists an element $x_0\in X$ with $x_0\leq f(x_0)$. Suppose that there exists $\varphi\in\Phi$ such that

$$\varphi(d(f(x),f(y))\leq\alpha(\varphi(d(x,y)))\varphi(d(y,v))$$

for all $x,y\in X$ and $y\leq v$ such that x and y are comparable and that $\alpha:[0,\infty)\to[0,1]$ is a function satisfying $\limsup_{s\to t^+}\alpha(s)<1$, for all $t\in(0,\infty)$. Assume that either f is continuous or X is such that the following holds: if an increasing sequence $\{x_n\}\to x$ in X, then $x_n\leq x$, for all $n\in\mathbb{N}$.
Then f has a fixed point.

2. Coupled Fixed Point Theorems

In this section we give the main results of our paper.

Theorem 2.1

Let $(X,d,\leq)$ be a partially ordered complete metric space, and let $F:X\times X\to X$ be a continuous mapping having the mixed monotone property on X. Suppose that there exists $\varphi\in\Phi$ such that

$$\varphi(d(F(x,y),F(u,v)))\leq\alpha(\varphi(d(x,u)))\alpha(\varphi(d(y,v)))\varphi(d(x,u)) \quad (1)$$

for all $x,y,u,v\in X$, where $x\geq u$ and $y\leq v$, and that $\alpha:[0,\infty)\to[0,1]$ is a function satisfying $\limsup_{s\to t^+}\alpha(s)<1$, for all $t\in(0,\infty)$. Assume that either X has the following property:

1. if an increasing sequence $\{x_n\}\to x$, then $x_n\leq x$, for all $n\in\mathbb{N}$.

If there exist $x_0,y_0\in X$ such that $x_0\leq F(x_0,y_0)$ and $y_0\geq F(x_0,y_0)$. Then there exist $x,y\in X$ such that

$$x=F(x,y) \quad \text{and} \quad y=F(y,x).$$

Proof. Since $x_0\leq F(x_0,y_0)$ and $y_0\geq F(x_0,y_0)$, then we suppose that $F(x_0,y_0)=x_1$, and $F(x_0,y_0)=y_1$. Now; by this assumption, put $x_2=F(x_1,y_1)$ and $y_2=F(y_1,x_1)$. We denote

$$F^2(x_0,y_0)=F(F(x_0,y_0),F(y_0,x_0))=F(x_1,y_1)=x_2,$$

and

$$F^2(y_0,x_0)=F(F(y_0,x_0),F(x_0,y_0))=F(y_1,x_1)=y_2.$$

Therefore by the mixed monotone property of F, we have

$$x_2=F^2(x_0,y_0)=F(x_1,y_1)\geq F(x_0,y_0)=x_1,$$

and

$$y_2=F^2(y_0,x_0)=F(y_1,x_1)\leq F(y_0,x_0)=y_1.$$

For $n=1,2,\cdots$, we let

$$x_{n+1}=F^{n+1}(x_0,y_0)=F(F^n(x_0,y_0),F^n(y_0,x_0)),$$

and

$$y_{n+1}=F^{n+1}(y_0,x_0)=F(F^n(y_0,x_0),F^n(x_0,y_0)).$$

By induction we obtain the following relations:

$$x_0\leq F(x_0,y_0)=x_1\leq F^2(x_0,y_0)=x_2\leq\ldots\leq F^{n+1}(x_0,y_0)\leq\cdots,$$

and

$$y_0\geq F(y_0,x_0)=y_1\geq F^2(y_0,x_0)=y_2\geq\ldots\geq F^{n+1}(y_0,x_0)\geq\cdots.$$

Then since $x_n\leq x_{n+1}$, and $y_n\geq y_{n+1}$, for each $n\in\mathbb{N}$, then by (2.1),

$$\varphi(d(F^{n+2}(x_0,y_0),F^{n+1}(x_0,y_0)))=\varphi(d(F(F^{n+1}(x_0,y_0),F^{n+1}(y_0,x_0)),F(F^n(x_0,y_0),F^n(y_0,x_0))))$$
$$\leq \alpha[\varphi(d(F^{n+1}(x_0,y_0),F^n(x_0,y_0)))]\alpha[\varphi(d(F^{n+1}(y_0,x_0),F^n(y_0,x_0)))]$$
$$\varphi(d(F^{n+1}(x_0,y_0),F^n(x_0,y_0)))$$
$$\leq \varphi(d(F^{n+1}(x_0,y_0),F^n(x_0,y_0))).$$

Similarly,

$$\varphi(d(F^{n+2}(y_0,x_0),F^{n+1}(y_0,x_0)))=\varphi(d(F(F^{n+1}(y_0,x_0),F^{n+1}(x_0,y_0)),F(F^n(y_0,x_0),F^n(x_0,y_0))))$$
$$=\varphi(d(F(F^n(y_0,x_0),F^n(x_0,y_0)),F(F^{n+1}(y_0,x_0),F^{n+1}(x_0,y_0))))$$
$$\leq \alpha[\varphi(F^n(y_0,x_0),F^{n+1}(y_0,x_0)))]\alpha[\varphi(d(F^n(x_0,y_0),F^{n+1}(x_0,y_0)))]$$
$$\varphi(d(F^n(y_0,x_0),F^{n+1}(y_0,x_0)))$$
$$\leq \varphi(d(F^n(y_0,x_0),F^{n+1}(y_0,x_0)))$$
$$=\varphi(d(F^{n+1}(y_0,x_0),F^n(y_0,x_0))).$$

Therefore $\{\varphi(d(F^{n+1}(x_0,y_0),F^n(x_0,y_0)))\}$ and $\{\varphi(d(F^{n+1}(y_0,x_0),F^n(y_0,x_0)))\}$ are decreasing sequences. These sequences are bounded below; thus

$$\lim_{n\to\infty}\varphi(d(F^{n+1}(x_0,y_0),F^n(x_0,y_0)))=r\geq 0,$$

and

$$\lim_{n\to\infty}\varphi(d(F^{n+1}(y_0,x_0),F^n(y_0,x_0)))=r'\geq 0.$$

Suppose that $r,r'>0$. By definition of α, we have $\alpha(r)<1$ and $\alpha(r')<1$. Therefore there exist $r_1,r_2\in[0,1)$, and $\varepsilon_1,\varepsilon_2>0$ such that for all $s\in[r,r+\varepsilon_1)$ and $s'\in[r',r'+\varepsilon_2)$, $\alpha(s)\leq r_1$ and $\alpha(s')\leq r_2$. By taking $n_0\in\mathbb{N}$ such that $r\leq\varphi(d(F^{n+2}(x_0,y_0),F^{n+1}(x_0,y_0)))\leq r+\varepsilon_1$, and $r'\leq\varphi(d(F^{n+2}(y_0,x_0),F^{n+1}(y_0,x_0)))\leq r'+\varepsilon_2$. Then

$$\varphi(d(F^{n+2}(x_0,y_0),F^{n+1}(x_0,y_0)))=\varphi(d(F(F^{n+1}(x_0,y_0),F^{n+1}(y_0,x_0)),F(F^n(x_0,y_0),F^n(y_0,x_0))))$$
$$\leq \alpha[\varphi(d(F^{n+1}(x_0,y_0),F^n(x_0,y_0)))]\alpha[\varphi(d(F^{n+1}(y_0,x_0),F^n(y_0,x_0)))]$$
$$\varphi(d(F^{n+1}(x_0,y_0),F^n(x_0,y_0)))$$
$$\leq r_1r_2\varphi(d(F^{n+1}(x_0,y_0),F^n(x_0,y_0))), \qquad (2)$$

and similarly

$$\varphi(d(F^{n+2}(y_0,x_0),F^{n+1}(y_0,x_0)))=\varphi(d(F(F^{n+1}(y_0,x_0),F^{n+1}(x_0,y_0)),F(F^n(y_0,x_0),F^n(x_0,y_0))))$$
$$=\varphi(d(F(F^n(y_0,x_0),F^n(x_0,y_0)),F(F^{n+1}(y_0,x_0),F^{n+1}(x_0,y_0))))$$
$$\leq \alpha[\varphi(F^n(y_0,x_0),F^{n+1}(y_0,x_0)))]\alpha[\varphi(d(F^n(x_0,y_0),F^{n+1}(x_0,y_0)))]$$
$$\varphi(d(F^n(y_0,x_0),F^{n+1}(y_0,x_0)))$$
$$\leq r_1r_2\varphi(d(F^{n+1}(y_0,x_0),F^n(y_0,x_0))) \qquad (3)$$

for every $n\geq n_0$. Thus, by above relations we have $r\leq r_1r_2r$ and $r'\leq r_1r_2r'$; these imply that $r=r'=0$ (note that since $r_1,r_2\in[0,1)$ therefore $r_1r_2\neq 1$).

It is clear that if $d(F^{m+2}(x_0,y_0),F^{m+1}(x_0,y_0))=0$ for some $m\in\mathbb{N}$, then $d(F^{n+2}(x_0,y_0),F^{n+1}(x_0,y_0))=0$ for $n\geq m$. This conclude that $\{F^n(x_0,y_0)\}$ is constant. Similarly for $\{F^n(y_0,x_0)\}$ we have it is constant. Thereupon F has a coupled fixed point.

Now; suppose that $d(F^{n+2}(x_0,y_0),F^{n+1}(x_0,y_0))\neq 0$ and $d(F^{n+2}(y_0,x_0),F^{n+1}(y_0,x_0))\neq 0$ for each $n\in\mathbb{N}$. Since $\varphi(d(F^{n+2}(x_0,y_0),F^{n+1}(x_0,y_0)))$ and $\varphi(d(F^{n+2}(y_0,x_0),F^{n+1}(y_0,x_0)))$ are decreasing and φ is increasing, then there exist nonnegative numbers u and v such that $d(F^{n+2}(x_0,y_0),F^{n+1}(x_0,y_0))$ and $d(F^{n+2}(y_0,x_0),F^{n+1}(y_0,x_0))$ converge to them, respectively. By conditions on φ and these sequences we can write $\varphi(u)\leq\varphi(d(F^{n+2}(x_0,y_0),F^{n+1}(x_0,y_0)))$ and $\varphi(v)\leq\varphi(d(F^{n+2}(y_0,x_0),F^{n+1}(y_0,x_0)))$, for every $n\in\mathbb{N}$. These imply

$$\varphi(u)\leq\lim_{n\to\infty}\varphi(d(F^{n+2}(x_0,y_0),F^{n+1}(x_0,y_0)))=r=0,$$

and

$$\varphi(v)\leq\lim_{n\to\infty}\varphi(d(F^{n+2}(y_0,x_0),F^{n+1}(y_0,x_0)))=r'=0.$$

Therefore $u=v=0$. By (2.2) and (2.3), we have

$$\sum_{n=1}^{\infty}\varphi(d(F^{n+1}(x_0,y_0),F^n(x_0,y_0)))\leq\sum_{n=1}^{n_0}\varphi(d(F^{n+1}(x_0,y_0),F^n(x_0,y_0)))+$$
$$\sum_{n_0+1}^{\infty}(r_1r_2)^n\varphi(d(F^{n_0+1}(x_0,y_0),F^{n_0}(x_0,y_0)))<\infty,$$

and

$$\sum_{n=1}^{\infty}\varphi(d(F^{n+1}(y_0,x_0),F^n(y_0,x_0)))\leq\sum_{n=1}^{n_0}\varphi(d(F^{n+1}(y_0,x_0),F^n(y_0,x_0)))+$$
$$\sum_{n_0+1}^{\infty}(r_1r_2)^n\varphi(d(F^{n_0+1}(y_0,x_0),F^{n_0}(y_0,x_0)))<\infty.$$

Since

$$\lim_{n\to\infty}\sup\frac{d(F^{n+1}(x_0,y_0),F^n(x_0,y_0))}{\varphi(d(F^{n+1}(x_0,y_0),F^n(x_0,y_0)))}\leq\lim_{s\to 0^+}\sup\frac{s}{\varphi(s)}<\infty,$$

and

$$\lim_{n\to\infty}\sup\frac{d(F^{n+1}(y_0,x_0),F^n(y_0,x_0))}{\varphi(d(F^{n+1}(y_0,x_0),F^n(y_0,x_0)))}\leq\lim_{s\to 0^+}\sup\frac{s}{\varphi(s)}<\infty,$$

then by *Limit Comparison Test Theorem*,

$$\sum_{n=1}^{\infty}d(F^{n+1}(x_0,y_0),F^n(x_0,y_0))<\infty,\quad\text{and}\quad\sum_{n=1}^{\infty}d(F^{n+1}(y_0,x_0),F^n(y_0,x_0))<\infty.$$

These mean that the sequences $\{F^n(x_0,y_0)\}$ and $\{F^n(y_0,x_0)\}$ are Cauchy sequences. Since (X,d) is a complete metric space, then there exist $x,y\in X$ such that

$$\lim_{n\to\infty}F^n(x_0,y_0)=x,\quad\text{and}\quad\lim_{n\to\infty}F^n(y_0,x_0)=y.$$

Finally, we claim $F(x,y)=x$ and $F(y,x)=y$. If (1) holds, then

$$\varphi(d(x,F(x,y)))\leq \lim_{n\to\infty}\inf \varphi(d(F^{n+1}(x_0,y_0),F(x,y)))$$
$$=\lim_{n\to\infty}\inf \varphi(d(F(F^n(x_0,y_0),F^n(y_0,x_0)),F(x,y)))$$
$$\leq \lim_{n\to\infty}\inf \alpha[\varphi(d(F^n(x_0,y_0),x))]\alpha[\varphi(d(F^n(y_0,x_0),y))]\varphi(d(F^n(x_0,y_0),x))$$
$$=0.$$

This implies that $F(x,y)=x$. Similarly we can show that $F(y,x)=y$.

Let $(X,\leq)$ be partially ordered set and suppose that there exists a metric d on X. Then the following statements are equivalent:

- every pair of elements has a lower bound or an upper bound;
- for every $x,y\in X$ there exists $z\in X$ which is comparable to x and y.

Theorem 2.2

In addition to the hypothesis of Theorem 2.1, suppose that every pair of X has an upper bound or a lower bound in X. Then $x=y$.

Proof. If x is comparable to y, then $x=F(x,y)$ is comparable to $y=F(y,x)$ and we get

$$\varphi(d(x,y))=\varphi(d(F(x,y),F(y,x)))\leq\alpha(\varphi(d(x,y))\alpha(\varphi(d(y,x))))\varphi(d(x,y))$$

this concludes that $\varphi(d(x,y))=0$ thus $x=y$.

If x is not comparable to y, then there exists an upper or lower bound of x and y. That is there exists a $z\varepsilon X$ comparable to x and y. Suppose that $x\leq z$, $y\leq z$ holds. Then we have $F(x,y)\leq F(z,y)$ and $F(x,y)\geq F(x,z)$, $F(y,x)\leq F(z,x)$ and $F(y,x)\geq F(y,z)$. Mixed monotone property of F yields that $F^{n+1}(x,y)\leq F^{n+1}(z,y)$, $F^{n+1}(x,y)\geq F^{n+1}(x,z)$, $F^{n+1}(y,x)\leq F^{n+1}(z,x)$ and $F^{n+1}(y,x)\geq F^{n+1}(y,z)$. These mean that $F^(x,z)=$ is comparable to $F^(z,x)=z,\ F^n(y,x)=y$ for $n=1,2,\ldots$. Then

$$\varphi(d(z,F^{n+1}(x,z)))=\varphi(d(F^{n+1}(z,x),F^{n+1}(x,z)))$$
$$=\varphi(d(F(F^n(z,x),F^n(x,z)),F(F^n(x,z),F^n(z,x))))$$
$$\leq\alpha[\varphi(d(F^n(z,x),F^n(x,z)))]\alpha[\varphi(d(F^n(x,z),F^n(z,x)))]\varphi(d(F^n(z,x),F^n(x,z)))$$
$$\leq\varphi(d(F^n(z,x),F^n(x,z)))=\varphi(d(z,F^n(x,z))). \quad (4)$$

Consequently, the sequence $\varphi(d(z,F^n(x,z)))=\varphi(d(F^{n+1}(z,x),F^{n+1}(x,z)))$ is a nonnegative decreasing sequence and hence possesses the limit γ. We claim that $\gamma=0$. Suppose that $\gamma>0$. Assume that

$$\limsup_{s\to\gamma^+}\alpha(s)\leq r_1 \quad \text{and} \quad \limsup_{t\to\gamma^+}\alpha(t)\leq r_2.$$

Then we have

$$\gamma\leq\limsup_{s\to\gamma^+}\alpha(s)\limsup_{t\to\gamma^+}\alpha(t)\gamma<R\gamma,$$

where $R=\max\{r_1,r_2\}$. This means that $\gamma=0$. In the next step, we should show that

$$\lim_{n\to\infty}(d(z,F^n(x,z)))=0$$

If $(d(z,F^m(x,z)))=0$ for some $m\in \mathbb{N}$, then $(d(z,F^n(x,z)))=0$ for all $n\geq m$. It follows that $\lim_{n\to\infty}(d(z,F^n(x,z)))=0$.

Suppose that $d(z,F^n(x,z))\neq 0$ for all $n\in \mathbb{N}$. Since $\varphi(d(z,F(x,z)))$ is a monotone decreasing sequence and φ is increasing, then $\{d(z,F^n(x,z))\}$ is a nonnegative and decreasing sequence and so $\lim_{n\to\infty}(d(z,F^n(x,z)))=r\geq 0$. But φ is lower semi continuous, then

$$\varphi(r)\leq \liminf_{n\to\infty} \varphi d(z,F^n(x,z))=\gamma=0$$

Hence r = 0. Analogously, it can be proved that $\lim_{n\to\infty}d(y,F^n(x,z))=0$. Finally,

$$d(z,y)\leq d(z,F^n(x,z))+d(F^n(x,z),y)$$

and taking as $n\to\infty$ yields $d(z,y)=0$

3.Application To Ordinary Differential Equations

Consider the periodic boundary value problem (PBVP)

$$\begin{cases} u'=h(t,u), & t\in I=[0,T]; \\ & (3) \\ u(0)=u(T). & \end{cases}$$

Where $T>0$ and $f:I\times \mathbb{R}\to \mathbb{R}$ is a continuous function. We assume that there exist continuous functions f,g such that

$$h(t,u)=f(t,u)+g(t,u),\quad t\in[0,T].$$

Existence of a unique solution to a periodic boundary value problem for mixed monotone mapping on partially ordered metric spaces was studied in [6]. Also, recently this equation is considered for single-valued mappings in [7], in case of generalization of Mizoguchi and Takahashi Theorem. In this section we study existence of solution to equation (3).

Consider the space $C(I,\mathbb{R})$ of continuous functions defined on $I=[0,T]$. Obviously, this space with the metric given by

$$d(x,y)=\sup\{|x(t)-y(t)|:t\in I\}\quad (x,y\in C(I)),$$

is a complete metric space. The metric space $C(I)$ can also equipped with a partial order given by

$$x,y\in C(I),\ x\leq y \Leftrightarrow x(t)\leq y(t)\quad (t\in I).$$

According to the Remark 3.3 of [6], every pair of elements of $(C(I,\mathbb{R}),\leq)$ has an upper bound or a lower bound in $(C(I,\mathbb{R}),\leq)$. Also, $C(I,\mathbb{R})\times C(I,\mathbb{R})$ is a partially ordered set if we define the following order relation in $C(I,\mathbb{R})$:

$$(x,y)\leq(u,v)\Leftrightarrow x(t)\leq u(t), y(t)\geq v(t)\quad (t\in I).$$

Also, $C(I,\mathbb{R})\times C(I,\mathbb{R})$ is complete metric space by following meter:

$$D((x,y),(u,v))=\sup_{t\in I}|x(t)-u(t)|+\sup_{t\in I}|y(t)-v(t)|\quad (x,y,u,v\in C(I,\mathbb{R})).$$

We recall the following definition from ([6, Definition 3.4]):

Definition 3.1

An element $(\alpha,\beta)\in X\times X$ *is called a coupled lower and upper solution of the PBVP (3) if*

$$\alpha'(t) \le f(t,\alpha(t)) + g(t,\beta(t)) \quad \text{and} \quad \beta'(t) \ge f(t,\beta(t)) + g(t,\alpha(t)),$$

together with the periodicity conditions,

$$\alpha(0) \le \alpha(T) \quad \text{and} \quad \beta(0) \ge \beta(T).$$

To prove of main result of this section we need to the following Lemma from [6]:

Lemma 3.2

If

$$\lambda_1(\alpha(T)-\alpha(0)) + \lambda_2(\beta(0)-\beta(T)) \le \frac{\alpha(T)-\alpha(0)}{T}; \qquad (5)$$

and

$$\lambda_1(\beta(0)-\beta(T)) + \lambda_2(\alpha(T)-\alpha(0)) \le \frac{\beta(0)-\beta(T)}{T}, \qquad (6)$$

then $\alpha(t) \le F(\alpha(t),\beta(t))$ and $\beta(t) \ge F(\beta(t),\alpha(t))$, for $t \in (0,T)$. Where

$$F(u,v)(t) = \int_0^T G_1(t,s)[f(s,u)+g(s,v)+\lambda_1 u - \lambda_2 v] + G_2(t,s)[f(s,v)+g(s,u)+\lambda_1 v - \lambda_2 u]ds. \qquad (7)$$

Now we ready to prove the following Theorem:

Theorem 3.3

Consider the problem (3) with $f, g \in C(I \times \mathrm{R}, \mathrm{R})$ *and suppose that there exist* $\lambda_1 > 0$ *and* $\lambda_2 > 0$ *such that for each* $u, v \in \mathrm{R}$ *with* $u \le v$

$$0 \le (f(s,v)-f(s,u)+\lambda_1(v-u)) + (g(s,v)-g(s,u)-\lambda_2(u-v))$$
$$\le (\lambda_1+\lambda_2)(\exp^{\log_{d(u,v)+1}^{\ln(d(v,u)+1)\ln(d(u,v)+1)}} - 1). \qquad (8)$$

Then the existence of a coupled lower and upper solution for (3), such (3.1) and (3.2) hold provides the existence of a unique solution of (3).

Proof. Problem (3) is equivalent to the integral equations

$$u(t) = \int_0^T G_1(t,s)[f(s,u)+g(s,v)+\lambda_1 u - \lambda_2 v] + G_2(t,s)[f(s,v)+g(s,u)+\lambda_1 v - \lambda_2 u]ds \qquad (9)$$

and

$$v(t) = \int_0^T G_1(t,s)[f(s,v)+g(s,u)+\lambda_1 v - \lambda_2 u] + G_2(t,s)[f(s,u)+g(s,v)+\lambda_1 u - \lambda_2 v]ds, \qquad (10)$$

where

$$G_1(t,s) = \begin{cases} \frac{1}{2}\left[\frac{e^{-(\lambda_1+\lambda_2)(t-s)}}{1-e^{-(\lambda_1+\lambda_2)T}} + \frac{e^{(\lambda_2-\lambda_1)(t-s)}}{1-e^{(\lambda_2-\lambda_1)T}}\right] & 0 \le s < t \le T \\ \frac{1}{2}\left[\frac{e^{-(\lambda_1+\lambda_2)(t-s+T)}}{1-e^{-(\lambda_1+\lambda_2)T}} + \frac{e^{(\lambda_2-\lambda_1)(t-s+T)}}{1-e^{(\lambda_2-\lambda_1)T}}\right] & 0 \le t < s \le T \end{cases}$$

and

$$G_2(t,s)=\begin{cases}\frac{1}{2}[\frac{e^{(\lambda_2-\lambda_1)(t-s)}}{1-e^{(\lambda_2-\lambda_1)T}}-\frac{e^{-(\lambda_1+\lambda_2)(t-s)}}{1-e^{-(\lambda_1+\lambda_2)T}}] & 0\le s<t\le T\\ \frac{1}{2}[\frac{e^{(\lambda_2-\lambda_1)(t-s+T)}}{1-e^{(\lambda_2-\lambda_1)T}}-\frac{e^{(\lambda_2-\lambda_1)(t-s+T)}}{1-e^{-(\lambda_1+\lambda_2)T}}] & 0\le t<s\le T\end{cases}$$

are Green functions (see [6]).

Define for $t\in I$

$$F(u,v)(t)=\int_0^T G_1(t,s)[f(s,u)+g(s,v)+\lambda_1 u-\lambda_2 v] + G_2(t,s)[f(s,v)+g(s,u)+\lambda_1 v-\lambda_2 u]ds. \quad (11)$$

Note that if (u,v) is a coupled fixed point of F, we have

$$u(t)=F(u,v)(t) \quad \text{and} \quad v(t)=F(v,u)(t),$$

for all $t\in I$. Now, we verify that F satisfies the hypotheses of Theorems 2.1 and 2.2. The mapping F having the mixed monotone property, because by (3.4), we have for $(u_1,v)\ge(u_2,v)$

$$\begin{aligned}F(u_1,v)(t)&=\int_0^T G_1(t,s)[f(s,u_1)+g(s,v)+\lambda_1 u_1-\lambda_2 v]\\&\quad+G_2(t,s)[f(s,v)+g(s,u_1)+\lambda_1 v-\lambda_2 u_1]ds\\&\ge\int_0^T G_1(t,s)[f(s,u_2)+g(s,v)+\lambda_1 u_2-\lambda_2 v]\\&\quad+G_2(t,s)[f(s,v)+g(s,u_2)+\lambda_1 v-\lambda_2 u_2]ds\\&=F(u_2,v)(t),\end{aligned}$$

and also for $(u,v_1)\le(u,v_2)$

$$\begin{aligned}F(u,v_1)(t)&=\int_0^T G_1(t,s)[f(s,u)+g(s,v_1)+\lambda_1 u-\lambda_2 v_1]\\&\quad+G_2(t,s)[f(s,v_1)+g(s,u)+\lambda_1 v_1-\lambda_2 u_1]ds\\&\ge\int_0^T G_1(t,s)[f(s,u)+g(s,v_2)+\lambda_1 u-\lambda_2 v_2]\\&\quad+G_2(t,s)[f(s,v_2)+g(s,u)+\lambda_1 v_2-\lambda_2 u]ds\\&=F(u_2,v)(t).\end{aligned}$$

Besides for $(x,y)\le(u,v)$,

$$\begin{aligned}\ln(d(F(u,v),F(x,y))+1)&=\ln(\sup_{t\in I}|F(u,v)(t)-F(x,y)(t)|+1)\\&=\ln(\sup_{t\in I}[\int_0^T G_1(t,s)([f(s,u)+g(s,v)+\lambda_1 u-\lambda_2 v]\\&\quad-[f(s,x)+g(s,y)+\lambda_1 x-\lambda_2 y])+G_2(t,s)([f(s,v)+g(s,u)+\lambda_1 v-\lambda_2 u]\\&\quad-[f(s,y)+g(s,x)+\lambda_1 y-\lambda_2 x])ds]+1)\\&=\ln(\sup_{t\in I}[\int_0^T G_1(t,s)([f(s,u)+g(s,v)+\lambda_1 u-\lambda_2 v]\\&\quad-[f(s,x)+g(s,y)+\lambda_1 x-\lambda_2 y])-G_2(t,s)([f(s,y)+g(s,x)+\lambda_1 y-\lambda_2 x]\\&\quad-[f(s,v)+g(s,u)+\lambda_1 v-\lambda_2 u])ds]+1)\\&\le\ln((\lambda_1+\lambda_2)(\exp^{\log_{d(y,v)+1}^{\ln(d(y,v)+1)\ln(d(u,x)+1)}}-1)\end{aligned}$$

$$\sup_{t\in I}[\int_0^T (G_1(t,s)-G_2(t,s))ds]+1)$$
$$= \ln((\lambda_1+\lambda_2)(\exp^{\log_{d(y,v)+1}^{\ln(d(y,v)+1)\ln(d(u,x)+1)}}-1)$$
$$\sup_{t\in I}[\int_0^t \frac{e^{-(\lambda_1+\lambda_2)(t-s)}}{1-e^{-(\lambda_1+\lambda_2)T}}ds+\int_t^T \frac{e^{-(\lambda_1+\lambda_2)(t-s+T)}}{1-e^{-(\lambda_1+\lambda_2)T}}ds]+1)$$
$$= \ln((\lambda_1+\lambda_2)(\exp^{\log_{d(y,v)+1}^{\ln(d(y,v)+1)\ln(d(u,x)+1)}}-1)\frac{1}{\lambda_1+\lambda_2}+1)$$
$$= \ln\exp^{\log_{d(y,v)+1}^{\ln(d(y,v)+1)\ln(d(u,x)+1)}} = \frac{\ln\ln(d(u,x)+1).\ln\ln(d(y,v)+1)}{\ln(d(y,v)+1)}$$
$$= \frac{\ln\ln(d(u,x)+1)}{\ln(d(u,x)+1)}.\frac{\ln\ln(d(y,v)+1)}{\ln(d(y,v)+1)}.\ln(d(u,x)+1)$$
$$= \alpha(\ln(d(u,x)+1))\alpha(\ln(d(y,v)+1))\ln(d(u,x)+1)$$

Put $\varphi(x)=\ln(x+1)$ and $\alpha(x)=\frac{\varphi(x)}{x}$. By definition, φ is a continuous, increasing and positive function in $(0,1)$, with $\varphi(0)=0$ and $\limsup_{s\to 0}\frac{s}{\varphi(s)}<\infty$. Also α is satisfies in condition of Theorem 2.1.

Finally, let (α,β) be a coupled upper and lower solution of equation (3). Then by Lemma 3.2, we have

$$\alpha(t)\le F(\alpha(t),\beta(t)) \quad \text{and} \quad \beta(t)\ge F(\beta(t),\alpha(t)).$$

Then by application of Theorems 2.1 and 2.2, proof is complete.

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